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Vacuum-polarization interaction energy between electrically neutral systems represented by Wilson loops

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Abstract. We analyse the vacuum-polarization interaction energy (VPIE) of electrically neutral systems represented by Wilson loops (WLs). Although small, the VPIE is a long-range interaction. This fact suggests that, in the absence of the classical dipole–dipole interaction, the cloud of virtual electron–positron pairs around one WL can disturb the other. Hence, the VPIE may be included in the physical explanation of forces, between neutral systems, with induced dipole behaviour. This VPIE is a static effect for separations beyond $2a_0$ (a_0 is the Bohr radius). Consequently, it is a complementary correction to the (retarded) Casimir–Polder atom–atom interaction which becomes important for separations beyond $137a_0$.

1. Introduction

The calculation of the interaction energy between colour singlet mesons, idealized as Wilson loops (WLs), has revealed the existence of long-range forces with induced dipole behaviour [1, 2] (van der Waals interaction). In the Abelian model, the interaction between the WL was exactly computed since all functional integrals that appear are of Gaussian type. The result, after subtracting the kinetic and self-energy of the sources, is the classical dipole–dipole interaction [1–3].

In this paper, we examine the contribution of the quantum electrical dynamical (QED) vacuum polarization to the interaction energy between electrically neutral systems represented by WLs. The correlation function of two WLs [4] is evaluated perturbatively, instead of using the usual approach where the WLs are expressed by external currents and Gaussian functional integrals are performed in order to obtain the classical result. With the perturbative technique, the results of [1–3] for the Abelian model are obtained by performing only ordinary integrals over a single photon exchange, since, after expressing the integration around the contour Γ_2 in terms of the integral around the contour Γ_1 (figure 1) by a convenient space-like translation, the perturbative series can be summed to all orders.

The perturbative approach is useful, since it enables us to include the photon self-energy correction. Hence, we can obtain the contribution of the vacuum polarization

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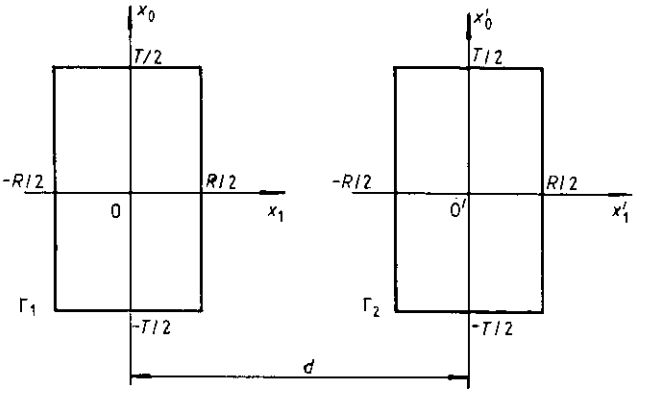

Figure 1. Rectangular WLs in the two-dimensional Euclidean subspace $\{e_0, e_1\}$ separated by a space-like distance d.

to the interaction energy, or vacuum-polarization interaction energy (VPIE), of two WLs separated by a space-like distance d.

The VPIE suggests that the QED vacuum polarization may be a crucial ingredient to be added to the physical explanation of long-range forces, with induced dipole behaviour, between neutral systems with a very symmetrical distribution of charges.

The outline of this work is as follows. In section 2 we compute perturbatively the correlation function of two WLs and the classical dipole–dipole interaction is obtained. The computation of the VPIE comprises section 3. The technique of dimensional regularization [5, 6] is used in order to cope with divergent integrals that arise in the algebraic treatment. Finally, in section 4, we present the concluding remarks.

2. Interaction energy

The interaction energy of two currents circulating around the rectangles Γ_1 and Γ_2 separated by a space-like distance d, as shown in figure 1, is given by the correlation function of two Wilson loops [4]

$$E(d) = -\lim_{T\to\infty} \frac{1}{T} \ln \left\{ \frac{\left\langle P\exp\left(\mathrm{ie}\oint_{\Gamma_1} A_\mu \mathrm{d}x^\mu\right) P\exp\left(\mathrm{ie}\oint_{\Gamma_2} A_\gamma \mathrm{d}x^\gamma\right)\right\rangle}{\left\langle P\exp\left(\mathrm{ie}\oint_{\Gamma_1} A_\mu \mathrm{d}x^\mu\right)\right\rangle \left\langle P\exp\left(\mathrm{ie}\oint_{\Gamma_2} A_\gamma \mathrm{d}x^\gamma\right)\right\rangle} \right\} \tag{2.1}$$

where the quantum average $\langle\ \rangle$ is given by the Euclidean generating functional of the quantized electromagnetic field

$$\langle \mathcal{O}(A_\mu)\rangle = \int D\left[A_\mu(x)\right] \exp\left(-\frac{1}{4}\int \mathrm{d}^\nu F^2_{\mu\gamma}\right) \mathcal{O}(A_\mu)$$

with $\mathcal{O}(A_\mu)$ denoting an observable. Since normalization is a matter of convention, by choosing an appropriate normalized functional measure $D\left[A_\mu(x)\right]$, we can write the average in the denominator of (2.1) equal to unity, that is

$$\left\langle P\exp\left(\mathrm{ie}\oint_{\Gamma_1} A_\mu \mathrm{d}x^\mu\right)\right\rangle \left\langle P\exp\left(\mathrm{ie}\oint_{\Gamma_2} A_\gamma \mathrm{d}x^\gamma\right)\right\rangle = 1$$

including gauge fixing terms.

The representation of electrically neutral systems by means of WLs is useful since (2.1) manifestly exhibits the gauge invariance of the interaction energy.

Usually, (2.1) is evaluated by expressing the WLs as external currents circulating around the rectangles of figure 1. With this approach the classical interaction energy can be exactly computed, since the functional integrals are of Gaussian type [1–3]. However as we will show here, with the perturbative technique we can include the photon self-energy correction and the tree approximation of the interaction energy can be obtained by performing only ordinary integrals over a single photon exchange.

The quantized WL, as defined by Wilson [7],

$$W(c) = \left\langle P e^{ie \oint A_\mu dx^\mu} \right\rangle \tag{2.2}$$

can be written as a perturbative series [8, 9]. Due to the Abelian character of the theory, the contribution to all orders can be summed [8, 10] and the tree approximation of (2.2) is given by

$$W_c(c) = \exp(W_2) \tag{2.3}$$

where the subscript c means classical or tree approximation. W_2 is the single photon exchange contribution,

$$W_2 = -\frac{e^2\mu^{(4-\nu)}}{2} \oint dx_\alpha \oint dy_\beta \int \frac{d^\nu k}{(2\pi)^\nu} e^{-ik(x-y)} \frac{\delta_{\alpha\beta}}{k^2} \tag{2.4}$$

where α and β are spacetime indices. The dimension of integration over k is given by ν and μ is an arbitrary mass parameter introduced in order to keep the charge dimensionless in ν dimensions [5, 6]. The factor $\frac{1}{2}$ arises because there are two equivalent choices of the origin in the parametrization of the contour.

The one-photon exchange contribution to (2.2) is given by W_2, (2.4), where we have two integrations around the same contour. But the one-photon exchange contribution to the interaction energy, (2.1), is given by $2W_2$ with one integration around each rectangle of figure 1. The factor two is introduced in order to cancel the factor $\frac{1}{2}$ of (2.4) because now we have only one choice of the origin in the parametrization of each contour. The rectangles of figure 1 are contained in a two-dimensional Euclidean subspace $\{e_0, e_1\}$ of the space time R^ν separated by a space-like distance d. The integration around the rectangle Γ_2 can be written in terms of the integration around the rectangle Γ_1 (figure 1) by performing a space-like translation $y_1 \to y_1 + d$, that is

$$\oint_{\Gamma_1} dx_0\, dx_1 \oint_{\Gamma_2} dy_0\, dy_1 \int \frac{d^\nu k}{(2\pi)^\nu} \frac{\exp\{-i|k|\left[(x_0-y_0)^2+(x_1-y_1)^2\right]^{1/2}\cos\theta\}}{k^2}$$

$$= \oint_{\Gamma_1} dx_0\, dx_1 \oint_{\Gamma_1} dy_0\, dy_1 \int \frac{d^\nu k}{(2\pi)^\nu} \frac{\exp\{-i|k|\left[(x_0-y_0)^2+(x_1-y_1-d)^2\right]^{1/2}\cos\theta\}}{k^2} \tag{2.5}$$

where θ is the angle between the vectors $(x-y)$ and k. Due to the Abelian character of the theory, the fourth-order contributions, that is the exchange of two photons, can

be written in terms of the one-photon exchange and the result is given by $(W_2)^2/2$. Moreover, the contribution to all orders can be written in terms of one-photon exchange, (2.3). Hence the tree approximation of the interaction energy, (2.1), can be written as

$$E_c(d) = -\lim_{T\to\infty} \frac{2}{T} W_2 \tag{2.6}$$

with $(y_1 + d)$ in place of (y_1) in (2.4). After the integration over the angular variables, (2.4) can be written in the form

$$W_2 = -\frac{e^2\mu^{2\epsilon}}{(2\pi)^\nu} \frac{\pi^{\nu/2}}{\Gamma(\nu/2)} \int_0^\infty \omega^{-2\epsilon} \sin(\omega)\, d\omega \oint_{\Gamma_1} dx_\alpha \oint_{\Gamma_1} dy_\alpha \frac{r^{2\epsilon}}{r^2} \tag{2.7}$$

where $\omega = |k|r$, $\epsilon = (4-\nu)/2$, and

$$r^2 = (x_0 - y_0)^2 + (x_1 - y_1 - d)^2 . \tag{2.8}$$

The remaining integrals of (2.7) are performed in appendix A and the dominant term, in the limit of $T \to \infty$, that is the tree approximation of the interaction energy given by the correlation function of two WLs, is obtained

$$\begin{aligned} E_c(d) &= -\frac{e^2}{8\pi} \left\{ (d+R)^{-1} + (d-R)^{-1} - 2d^{-1} \right\} \\ &= -\frac{e^2}{4\pi} \sum_{N=1}^{\infty} \frac{R^{2N}}{d^{(2N+1)}} \end{aligned} \tag{2.9}$$

where R is the space-like size of the rectangles of figure 1 and d is the space-like distance between them. Of course, we can write an arbitrary charge $z'z''e^2$, with z' and z'' integers, in place of e^2, since (2.3) is obtained by summing the perturbative series to all orders. We reproduced the result achieved by the approach where the WLs are expressed by means of external currents and the functional integrals are of Gaussian type [1–3]. The perturbative treatment presented here is useful, since it enables us to include, in the next section, the photon self-energy correction.

3. Vacuum-polarization contribution

If we include the photon self-energy correction, only one internal loop per exchanged photon, the perturbative series can be summed due to the Abelian properties of the theory. Hence, we obtain the one-loop approximation for (2.2)

$$\begin{aligned} W(c) &= 1 + (W_2 + \overline{W}_2) + \frac{1}{2!}(W_2 + \overline{W}_1)^2 + \frac{1}{3!}(W_2 + \overline{W}_2)^3 + \cdots \\ &= \exp(W_2 + \overline{W}_2) \end{aligned} \tag{3.1}$$

where $\overline{W}_2$ is the fourth-order contribution due to the insertion of the self-energy correction in a single photon exchange. The integral over k can be carried out by

introducing polar coordinates in ν dimensions. After performing the integral over the internal fermionic loop and over the angular variables we have [5, 6, 11]

$$\overline{W}_2 = -\frac{e^2}{2}\left\{\frac{e^2\mu^{2(4-\nu)}}{(2\pi)^{2\nu}\,(m^2)^{(4-\nu)/2}}\,\frac{\pi^{\nu/2}}{3}\mathrm{d}(\nu)\right\}\Gamma\left(\frac{4-\nu}{2}\right)\oint_{\Gamma_1}\mathrm{d}x_\alpha\oint_{\Gamma_1}\mathrm{d}y_\alpha\int\mathrm{d}^\nu k$$
$$\times\exp\{-\mathrm{i}k\left[(x_0-y_0)^2+(x_1-y_1-d)^2\right]^{1/2}\}\frac{1}{k^2}F\left(\frac{4-\nu}{2},2;\frac{5}{2};-\frac{k^2}{4m^2}\right) \tag{3.2}$$

where m is the mass of the electron. From (2.1), (3.1) and (3.2) we obtain the vacuum-polarization contribution for the interaction energy of the WLs of figure 1

$$\overline{E}(d) = \mathcal{C}(\epsilon)\frac{3}{2}\int_0^1\left[(1-t)^{-1/2}-(1-t)^{1/2}\right]\mathrm{d}t\int_0^\infty x^{-2\epsilon}\sin(x)\mathrm{d}x$$
$$\times\lim_{T\to\infty}\frac{1}{T}\oint_{\Gamma_1}\mathrm{d}x_\alpha\oint_{\Gamma_1}\mathrm{d}y_\alpha\left[r^2\left(\frac{1}{m^2r^2}+\frac{x^2t}{4m^4r^4}\right)^\epsilon\right]^{-1}. \tag{3.3}$$

The parameter $\mathcal{C}(\epsilon)$ is given by

$$\mathcal{C}(\epsilon) = \frac{e^2}{3}\,\frac{\alpha}{8\pi^3}\left[-\frac{1}{\epsilon}-1+2\gamma+\ln(2)+2\ln\left(\frac{m^2}{4\pi\mu^2}\right)\right]+\mathcal{O}(\epsilon) \tag{3.4}$$

where γ is the Euler's constant [12, 13], α is the fine structure constant, and $\epsilon = (4-\nu)/2$.

From appendix B we have the finite part of the vacuum-polarization contribution for the interaction energy [14]

$$-\frac{\lambda\!\!\!^{-}_e\overline{E}(d)}{e^2} = \frac{\alpha}{3\pi}\frac{1}{8\pi}\left[\frac{1}{3}\left(\frac{\lambda\!\!\!^{-}_e}{d-R}-2\frac{\lambda\!\!\!^{-}_e}{d}+\frac{\lambda\!\!\!^{-}_e}{d+R}\right)\right.$$
$$\left.+\frac{3}{4}\sum_{l=1}^{3}A_l\int_1^\infty\left(\frac{1}{u^2}+\frac{1}{u^4}\right)\mathrm{d}u\left\{\mathrm{e}^{-2B_lu}E_i(2B_lu)+\mathrm{e}^{2B_lu}E_i(-2B_lu)\right\}\right] \tag{3.5}$$

where $\lambda\!\!\!^{-}_e$ is the reduced electron Compton wavelength. In order to perform the numerical calculation of table 1 and the plots of (3.5) and (3.6) depicted in figure 2 we take $R = 2a_0$, where a_0 is the Bohr radius.

The behaviour of the remaining integral,

$$F\left(\frac{d}{\lambda\!\!\!^{-}_e}\right) = \frac{\alpha}{3\pi}\,\frac{1}{8\pi}\times\frac{3}{4}\sum_{l=1}^{3}A_l\int_1^\infty\left(\frac{1}{u^2}+\frac{1}{u^4}\right)\mathrm{d}u\left\{\mathrm{e}^{-2B_lu}E_i(2B_lu)+\mathrm{e}^{2B_lu}E_i(-2B_lu)\right\} \tag{3.6}$$

reflects, essentially, the overlap of the clouds with a non-uniform distribution of virtual charges around the WLs of figure 1. As shown in table 1, the contribution of (3.6) for $l = 2, 3$, that is $F_2(d/\lambda\!\!\!^{-}_e)$ and $F_3(d/\lambda\!\!\!^{-}_e)$, are small when confronted with that of $l = 1$, that is $F_1(d/\lambda\!\!\!^{-}_e)$. From the confrontation of (3.5) with the classical result, (2.9), we obtain the order of magnitude of the vacuum-polarization contribution to the interaction energy

$$\overline{E}\left(\frac{d}{\lambda\!\!\!^{-}_e}\right)\approx\frac{\alpha}{9\pi}E_c\left(\frac{d}{\lambda\!\!\!^{-}_e}\right). \tag{3.7}$$

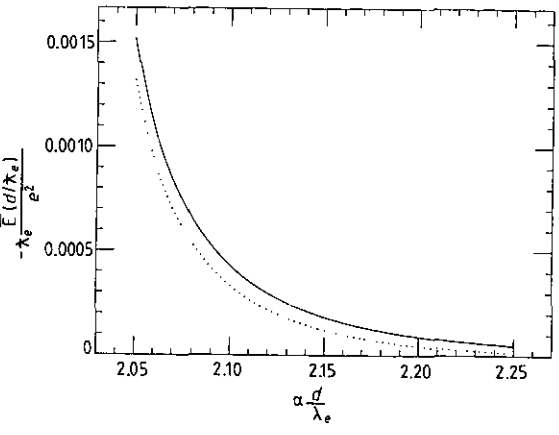

Figure 2. The vacuum-polarization contribution to the interaction energy between the WLs of figure 1 is given by the full line. The dotted line is the contribution of (3.6).

Table 1. Numerical values of (3.6) for $l = 1$ ($F_1(d/\lambda\!\!\!^{-}_e)$), $l = 2$ ($F_2(d/\lambda\!\!\!^{-}_e)$), $l = 3$ ($F_3(d/\lambda\!\!\!^{-}_e)$), sum of the three contributions ($F(d/\lambda\!\!\!^{-}_e)$) multiplied by $(\alpha/3\pi)(8\pi)^{-1}$ and the vacuum-polarization interaction energy, (3.5). In the numerical calculation we assumed $R = 2\,a_0$ (a_0 is the Bohr radius).

$\frac{d}{\lambda\!\!\!^{-}_e}$	$F_1\left(\frac{d}{\lambda\!\!\!^{-}_e}\right)$	$F_2\left(\frac{d}{\lambda\!\!\!^{-}_e}\right)$	$F_3\left(\frac{d}{\lambda\!\!\!^{-}_e}\right)$	$F\left(\frac{d}{\lambda\!\!\!^{-}_e}\right)$	$-\lambda\!\!\!^{-}_e E\left(\frac{d}{\lambda\!\!\!^{-}_e}\right)/e^2$
2.005	9.89043045	0.12154372	−0.01350834	0.00061605	0.12597206
2.010	5.76897526	0.12093722	−0.01347181	0.00036208	0.02758651
2.015	4.51249504	0.12033440	−0.01343544	0.00028462	0.01098269
2.020	3.84101343	0.11973524	−0.01339922	0.00024321	0.01284789
2.030	2.03807998	0.11854818	−0.01332720	0.00010367	0.00691238
2.040	2.52124143	0.11737574	−0.01325580	0.00016176	0.00438528
2.050	2.14104557	0.11621786	−0.01318496	0.00013827	0.00303742
2.060	1.84339893	0.11507421	−0.01311471	0.00011986	0.00222362
2.070	1.60160017	0.11394464	−0.01304503	0.00010490	0.00169151
2.080	1.40019584	0.11282890	−0.01297621	0.00009243	0.00132354
2.090	1.22935176	0.11172714	−0.01290765	0.00008183	0.00105827
2.100	1.08239901	0.11063877	−0.01283966	0.00007272	0.00086078
2.150	0.57641751	0.10539469	−0.01250774	0.00004124	0.00036236
2.200	0.28442249	0.10046413	−0.01218884	0.00004124	0.00017916
2.250	0.10199138	0.09582741	−0.01188226	0.00001146	0.00009632
2.300	−0.01648333	0.09146560	−0.01158748	0.00000391	0.00005428
2.350	−0.09468444	0.08736135	−0.01130373	−0.00000115	0.00003138
2.400	−0.14628029	0.08349725	−0.01103053	−0.00000455	0.00001835
2.450	−0.17972815	0.07985816	−0.01076740	−0.00000682	0.00001074
2.500	−0.20053181	0.07642934	−0.01051412	−0.00000829	0.00000624
2.550	−0.21240641	0.07319715	−0.01026962	−0.00000921	0.00000358
2.600	−0.21702379	0.07014892	−0.01003382	−0.00000972	0.00000205
2.650	−0.21891457	0.06727245	−0.00980625	−0.00000995	0.00000120
2.700	−0.21668789	0.06455681	−0.00958671	−0.00000996	0.00000077
2.750	−0.21220310	0.06199147	−0.00937453	−0.00000983	0.00000061
2.800	−0.20615986	0.05956718	−0.00916953	−0.00000960	0.00000060

4. Concluding remarks

The VPIE, equation (3.7), and the Casimir–Polder potential (CPP) [15–19] seem to have quite different physical meanings, since they behave asymptotically with different powers of $1/d$. It is worth recalling that classical 'atoms' do not attract one another at separations beyond a_0 (a_0 is the Bohr radius), the (non-relativistic) van der Waals d^{-6} interaction can be thought of as being a consequence of the uncertainty principle for particles. In the (retarded) Casimir–Polder d^{-7} atom–atom interaction, the crucial ingredient is the retardation time—the transit time of a photon—which occurs for d larger than about $137\,a_0$. The CPP can be thought of as having its origins in QED vacuum fluctuations, that is in the uncertainty principle for electromagnetic fields. On the other hand, the VPIE given by (3.7) is a static effect which occurs for separation beyond $2\,a_0$. Hence it is a complementary correction to the CPP and the dominant contribution for $d > R = 2\,a_0$ is proportional to a multipole expansion

$$-\lambda\!\!\!^{-}_e \overline{E}\left(\frac{d}{\lambda\!\!\!^{-}_e}\right)/e^2 \approx \frac{\alpha}{72\pi^2}\left(\frac{\lambda\!\!\!^{-}_e}{d-R} - 2\frac{\lambda\!\!\!^{-}_e}{R} + \frac{\lambda\!\!\!^{-}_e}{d+R}\right) = \frac{\alpha\lambda\!\!\!^{-}_e}{36\pi^2}\sum_{N=1}^{\infty}\frac{R^{2N}}{d^{(2N+1)}}. \tag{4.1}$$

From this equation we readily observe that the term in the asymptotic limit for a very large value of d is proportional to the classical dipole–dipole interaction. Furthermore, the interaction is attractive since the dipolar moments of the systems analysed are parallel.

Atoms are of course not static so that (3.7) does not entirely describe the vacuum contribution to the interaction energy, for example, of two helium atoms. If the two atoms, sufficiently far apart so that their wavefunctions do not overlap, are in their ground states, the dipole (and all higher-order multipole) moments of each atom vanish, and therefore the clouds of electrons have a very symmetrical distribution of charges. But, a non-vanishing interaction is obtained from the second order dipole–dipole perturbation [20].

In this context, we stress the fact that (4.1) gives a dipole–dipole term of order α for large values of d, that is $d >> R = 2\,a_0$. This means that, if we discard the permanent interaction given by (2.9), the cloud of virtual electron–positron pairs around one WL, figure 1, can disturb the other. Hence, the VPIE is capable of inducing a dipole moment between them.

Although the representation of neutral systems by WLs is an extremely simple model, (4.1) leads to the conclusion that the QED vacuum polarization may be significant in the appearance of induced dipole moments. In that case, it must be added to the physical explanation of long-range forces, with induced dipole behaviour (van der Waals interactions), among neutral systems with a very symmetrical distribution of charges.

For a non-Abelian theory, the exchange of two or more photons cannot be written in terms of one-photon exchange. Hence, the tree approximation contributions to all orders to (2.1) cannot be summed.

The existence of van der Waals interactions in QCD for small coupling, [1, 2] due to colour effects, has emerged even for static mesons in the tree approximation. But before addressing the possible phenomenological consequences of van der Waals forces in QCD, we need a strong coupling calculation where confinement is taken into account *ab initio*.

The vacuum-polarization effect in a non-Abelian theory, for example QCD, should be taken into account after a complete analysis of the tree approximation for any arbitrary value of the coupling.

Appendix A. Integration over x_α and y_α around the rectangle Γ_1

In the limit $\nu \to 4$, that is for small ϵ, we can write (2.7) in the form

$$W_2 = -\frac{e^2}{2^4\pi^2}\int_0^\infty \omega^{-2\epsilon}\sin(\omega)\mathrm{d}\omega \oint \mathrm{d}y_\alpha \oint \mathrm{d}y_\alpha \frac{1}{r^2} + \mathcal{O}(\epsilon) \tag{A.1}$$

where in a two-dimensional space $\{e_0, e_1\}$, that is around the contour Γ_1 ($\alpha = 0, 1$) of figure 1, we have

$$r^2 = (x_0 - y_0)^2 + (x_1 - y_1 - d)^2 \tag{A.2}$$

and the integration over x_α and y_α are given by

$$\begin{aligned} I = \oint_{\Gamma_1} \mathrm{d}y_\alpha \oint_{\Gamma_1} \mathrm{d}x_\alpha & \frac{1}{(x_0 - y_0)^2 + (x_1 - x_1 - d)^2} \\ = \oint_{\Gamma_1} \mathrm{d}y_\alpha \Bigg\{ & \int_{-T/2}^{T/2} \frac{\mathrm{d}x_0}{(x_0 - y_0)^2 + \left(\frac{R}{2} - y_1 - d\right)^2} \\ & - \int_{-T/2}^{T/2} \frac{\mathrm{d}x_0}{(x_0 - y_0)^2 + \left(-\frac{R}{2} - y_1 - d\right)^2} \\ & + \int_{-R/2}^{R/2} \frac{\mathrm{d}x_1}{\left(-\frac{T}{2} - y_0\right)^2 + (x_1 - y_1 - d)^2} \\ & - \int_{-R/2}^{R/2} \frac{\mathrm{d}x_1}{\left(\frac{T}{2} - y_0\right)^2 + (x_1 - y_1 - d)^2} \Bigg\}. \end{aligned} \tag{A.3}$$

After performing all the integrals over x_0 and x_1, we obtain

$$\begin{aligned} I = \oint_{\Gamma_1} \mathrm{d}y_\alpha \Bigg\{ & -\frac{1}{\left(y_1 + d - \frac{R}{2}\right)} \arctan\left(\frac{y_0 - \frac{T}{2}}{y_1 + d - \frac{R}{2}}\right) \\ & + \frac{1}{\left(y_1 + d - \frac{R}{2}\right)} \arctan\left(\frac{y_0 + \frac{T}{2}}{y_1 + d - \frac{R}{2}}\right) \\ & + \frac{1}{\left(y_1 + d + \frac{R}{2}\right)} \arctan\left(\frac{y_0 - \frac{T}{2}}{y_1 + d + \frac{R}{2}}\right) \end{aligned}$$

$$- \frac{1}{(y_1 + d + \frac{R}{2})} \arctan\left(\frac{y_0 + \frac{T}{2}}{y_1 + d + \frac{R}{2}}\right)$$
$$+ \frac{1}{(y_0 - \frac{T}{2})} \arctan\left(\frac{y_1 + d - \frac{R}{2}}{y_0 - \frac{T}{2}}\right) - \frac{1}{(y_0 - \frac{T}{2})} \arctan\left(\frac{y_1 + d + \frac{R}{2}}{y_0 - \frac{T}{2}}\right)$$
$$- \frac{1}{(y_0 + \frac{T}{2})} \arctan\left(\frac{y_1 + d - \frac{R}{2}}{y_0 + \frac{T}{2}}\right) + \frac{1}{(y_0 + \frac{T}{2})} \arctan\left(\frac{y_1 + d + \frac{R}{2}}{y_0 + \frac{T}{2}}\right)\Bigg\}. \tag{A.4}$$

We have two integrals over y_1, one for the path $y_0 = (T/2) + \delta$ and one other for the path $y_0 = -(T/2) - \delta$, where δ is a small constant. Half of the terms in the integrand of the sum of these two integrals cancel and the remaining terms are given by

$$2 \int_{-R/2}^{R/2} \mathrm{d}y_1 \Bigg\{ -\frac{1}{\delta} \arctan\left(\frac{y_1 + d - \frac{R}{2}}{\delta}\right) + \frac{1}{\delta} \arctan\left(\frac{y_1 + d + \frac{R}{2}}{\delta}\right)$$
$$+ \frac{1}{T + \delta} \arctan\left(\frac{y_1 + d - \frac{R}{2}}{T + \delta}\right) - \frac{1}{T + \delta} \arctan\left(\frac{y_1 + d + \frac{R}{2}}{T + \delta}\right)\Bigg\}. \tag{A.5}$$

This integral gives no contribution for large values of T, that is

$$\lim_{T \to \infty} \frac{1}{T} \left(\int_{-R/2}^{R/2} \mathrm{d}y_1 \{ \quad \} \right) = 0. \tag{A.6}$$

Finally, we perform the integrals over y_0 for the paths $y_1 = \pm R/2$. The non-null contribution in the limit of very large T, that is

$$\lim_{T \to \infty} \frac{1}{T} \left(\int_{-T/2}^{T/2} \mathrm{d}y_0 \{ \quad \} \right) \neq 0 \tag{A.7}$$

is given by

$$\oint_{\Gamma_1} \mathrm{d}y_\alpha \oint_{\Gamma_1} \mathrm{d}x_\alpha \frac{1}{(x_0 - y_0)^2 + (x_1 - y_1 - d)^2}$$
$$= -\frac{2T}{d + R} \arctan\left(\frac{T}{d + R}\right) + \frac{4T}{d} \arctan\left(\frac{T}{d}\right) - \frac{2T}{d - R} \arctan\left(\frac{T}{d - R}\right)$$
$$+ (\text{terms that give no contribution to the interaction energy for large values of } T). \tag{A.8}$$

Since

$$\int_0^\infty \omega^{-2\omega} \sin(\omega) \mathrm{d}\omega = \Gamma(2\epsilon + 1) \sin\left[(1 - 2\epsilon)\frac{\Gamma}{2}\right] = 1 + \mathcal{O}(\epsilon)$$

we obtain the tree approximation of the interaction energy given by (2.1)

$$E(d) = -\lim_{T\to\infty}\frac{2}{T}W_2 = -\lim_{T\to\infty}\frac{2}{T}\left\{\frac{e^2}{2^4\pi^2}2T\left[(d+R)^{-1}\arctan\left(\frac{T}{d+R}\right)\right.\right.$$
$$\left.\left. - 2d^{-1}\arctan\left(\frac{T}{d}\right) + (d-R)^{-1}\arctan\left(\frac{T}{d-R}\right)\right]\right\}$$

that is

$$E(d) = -\frac{e^2}{8\pi}\left\{(d+R)^{-1} - 2d^{-1} + (d-R)^{-1}\right\}. \tag{A.9}$$

Appendix B

In the limit of small ϵ, that is in the limit $\nu \to 4$, the integrand of (3.3) can be written in the form

$$\left(r^2\left(\frac{1}{m^2r^2} + \frac{x^2t}{4m^2r^4}\right)^{\epsilon}\right)^{-1}$$
$$= \frac{1}{r^2}\left[1 + \left(\int \mathrm{d}z\left\{\frac{\mathrm{d}}{\mathrm{d}z}\left(\left(\frac{z^2}{m^2r^2} + \frac{x^2t}{4m^4r^2}\right)^{\epsilon}\right)^{-1}\right\}\right)\Bigg|_{z=1}\right] \tag{B.1}$$
$$= \frac{1}{r^2} - 2\epsilon\left(\int\frac{\mathrm{d}z}{z}\left(r^2 + \frac{x^2t}{4m^2z^2}\right)^{-1}\right)\Bigg|_{z=1} + \mathcal{O}(\epsilon^2)$$

where we take the value $z = 1$ after performing the integral over z.

Substituting (B.1) into (3.3), for small ϵ, gives

$$\overline{E}(d) = \mathcal{C}(\epsilon)\frac{3}{2}\int_0^1\left[(1-t)^{-1/2} - (1-t)^{1/2}\right]\mathrm{d}t\left\{\int\frac{\mathrm{d}z}{z}\int_0^\infty x^{-2\epsilon}\sin(x)\mathrm{d}x\right.$$
$$\left.\times\lim_{T\to\infty}\frac{1}{T}\oint_{\Gamma_1}\mathrm{d}x_\alpha\oint_{\Gamma_1}\mathrm{d}y_\alpha\left[\frac{z}{r^2} - 2\epsilon\left(r^2 + \frac{x^2t}{4m^2z^2}\right)^{-1}\right]\right\}_{z=1} + \mathcal{O}(\epsilon) \tag{B.2}$$

where $r^2 = (x_0 - y_0)^2 + (x_1 - y_1 - d)^2$ and the index $\alpha = 0, 1$.

The integration around the contour Γ_1 of figure 1, see appendix A, gives

$$\overline{E}(d) = 2\mathcal{C}(\epsilon)\sum_{l=1}^{3}\left\{-\pi\left(1 + \frac{2}{\epsilon}\right)A_l + \epsilon\frac{3\pi}{4}\int_0^1\left(t^{-1/2} + t^{1/2}\right)\mathrm{d}t\right.$$
$$\left.\times\int\frac{\mathrm{d}z}{z}A_l\int_0^\infty x^{-2\epsilon}\sin(x)\left(1 + \frac{x^2t}{B_l^2z^2}\right)^{-1}\mathrm{d}x\right\}\Bigg|_{z=1} + \mathcal{O}(\epsilon) \tag{B.3}$$

where we used elementary properties of the hypergeoemtric function [13] in order to rewrite the integral over t. The terms with the index $l = 1, 2, 3$ are given by

$$A_1 = \frac{\lambda\!\!\!^{-}_e}{d-R} \qquad A_2 = -2\frac{\lambda\!\!\!^{-}_e}{d} \qquad A_3 = \frac{\lambda\!\!\!^{-}_e}{d+R}$$

and

$$B_1 = \frac{d-R}{\lambda\!\!\!^{-}_e} \qquad B_2 = \frac{d}{\lambda\!\!\!^{-}_e} \qquad B_3 = \frac{d+R}{\lambda\!\!\!^{-}_e}$$

where $\lambda\!\!\!^{-}_e$ is the reduced electron Compton wavelength.

After a change of variable $t^{-1/2} \to u$ and to perform the integrations over x and z, we obtain the interaction energy

$$-\frac{\lambda\!\!\!^{-}_e \overline{E}(d)}{e^2} = \frac{\alpha}{3\pi}\frac{1}{8\pi}\sum_{l=1}^{3}\left[\left(-\frac{1}{\epsilon} - 1 + \ln\left(\frac{m^2}{4\pi\mu^2}\right)\right)A_l + \frac{3}{4}A_l\int_0^\infty\left(\frac{1}{u^2}+\frac{1}{u^4}\right)\mathrm{d}u\right.$$

$$\left.\times\left\{\mathrm{e}^{-2B_l u}E_i(2B_l u) + \mathrm{e}^{2B_l u}E_i(-2B_l u)\right\}\right] \tag{B.4}$$

where $E_i(\pm\ \ )$ is the exponential integral function [12].

In order to obtain the finite part of the VPIE, (B.4), we need to cope with arbitrariness inherent in the renormalization [21]. The minimal substraction (MS) scheme has become standard specially in works on QCD. The disadvantage is that the MS scheme tends to produce large coefficients in the perturbation expansion. On the other hand, if the coefficient

$$-\frac{1}{\epsilon} - 1 + \ln(2) + \ln\left(\frac{m^2}{4\pi\mu^2}\right)$$

is smaller than 0.2 the VPIE becomes repulsive for large values of $(d-R)/\lambda\!\!\!^{-}_e$. But the dipolar moments of the system analysed are parallel and the interaction must be attractive. Hence we have a reasonable lower bound for this coefficient. In order to perform the numerical calculation we take the value 1/3 after to discard the pole $1/\epsilon$.

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